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Iteration Schemes for Solving Rectangular Games

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1. Introduction. In connection with linear programming problems, or in the consideration of zero-sum two person games in normal form the problem arises of computing the value of and optimal strategies for a rectangular game given by an $m \times n$ matrix $A = (a_{ij})$. Various iteration schemes have been proposed for achieving this and several have been proved to "converge" (i.e. accomplish the desired purpose). These schemes vary in the degree of their complexity, and with regard to what assertions can be made a priori concerning the "rate of convergence". For example, Von Neumann [1] has proposed an iteration scheme which though cumbersome for hand computations is easily coded for machine work, and for which he proves a very good a priori estimate of the rate of convergence. On the other hand, the scheme of George Brown and Julia Robinson [2] is easily carried out without a computing machine, and has also been proved to converge In this note it is proposed to consider this latter scheme and by suitably modifying and elaborating upon the method of [2] provide for it an a priori estimate of the rate of convergence.

2. Description of the iteration scheme, and preliminary lemmas. The iteration scheme of George Brown and Julia Robinson is based upon the following procedure of fictitious play. Given the $r \times x$ game matrix $A = (a_{ij})$, we start with two initial vectors $U(0) = (U_1(0), \ldots, U_r(0))$, and $V(0) = (V_1(0), \ldots, V_s(0))$. $U_i(0)$ is player I's initial relative estimate of what he expects to win

should he play the j-th column. Player I then chooses the row $i(0)$ which is such that $U_{i(0)} = \max U(0)$, ($\max U(0)$ denotes the maximum component of $U(0)$), and player II chooses the $j(0)$ column wherein $V_{j(0)} = \min V(0)$. That is, player I picks the row from which he expects to get the most and player II picks the column which he expects will give player I the least. The two players then revise their estimates of the relative values of the rows and columns to:

$$V(1) = V(0) + R_{i(0)} \quad ,$$

$$U(1) = U(0) + C_{j(0)} \quad ,$$

(where R_i denotes the i-th row of A and C_j denotes the j-th column of A). Then for the second play players I and II pick the $i(1)$ row and $j(1)$ column respectively, where $U_{i(1)}(1)=\max U(1)$ and $V_{j(1)}(1) = \min V(1)$. As the play proceed in this way we have

(2.1) $$U(n+1) = U(n) + C_{j(n)} \quad ,$$

(2.2) $$V(n+1) = V(n) + R_{i(n)} \quad ,$$

where

$$V_{j(n)}(n) = \min V(n) \quad ,$$

and

$$U_{i(n)}(n) = \max U(n) \quad .$$

We then obtain that

(2.3) $$U(n) = U(0) + \sum_{k=1}^{s} \sigma_k(n) C_k \quad ,$$

$$(2.4) \qquad V(n) = V(0) + \sum_{k=1}^{r} \tau_k(n) R_k \quad ,$$

where $\tau_k(n)$ equals the number of solutions of $j(\ell) = k$, $0 \le \ell \le n-1$; and $\tau_k(n)$ equals the number of solutions of $i(\ell)=k$, $0 \le \ell \le n-1$. Clearly $\sum_{k=1}^{s} \sigma_k(n) = n$ and $\sum_{k=1}^{r} \tau_k(n) = n$, so that dividing (2.3) and (2.4) by n we obtain

$$(2.5) \qquad \frac{U(n)}{n} = \frac{U(0)}{n} + \sum_{k=1}^{s} \frac{\sigma_k(n)}{n} C_k \quad ,$$

and

$$(2.6) \qquad \frac{V(n)}{n} = \frac{V(0)}{n} + \sum_{k=1}^{r} \frac{\tau_k(n)}{n} R_k \quad ,$$

where $\sigma(n) = (\sigma_1(n)/n, \ldots, \sigma_s(n)/n)$ and $\tau(n) = (\tau_1(n)/n, \ldots, \tau_r(n)/n)$ are strategy vectors.

We next prove several lemmas which are needed for later arguments.

Lemma 2.1. If v is the value of the game matrix A then

$$(2.7) \qquad \frac{\max U(n) - \min U(0)}{n} \ge v \ge \frac{\min V(n) - \max V(0)}{n} \quad .$$

Proof: Letting $y_k(n) = \sigma_k(n)/n$ we have from (2.5)

$$\frac{\max U(n)}{n} \ge \frac{\min U(0)}{n} + \max \left(\sum_{k=1}^{n} y_k(n) C_k\right) \quad ,$$

and since

$$\max \left(\sum_{k=1}^{s} y_k(n) C_k\right) = \max_i \left(\sum_{k=1}^{r} a_{ik} y_k(n)\right) \ge v \quad ,$$

the left inequality in (2.7) follows. The right inequality may be deduced similarly from (2.6).

For convenience we introduce the following notations:

$$\Delta_{U,V}(n) = \max U(n) - \min V(n) \quad ,$$

$$\Delta_{V,U}(n) = \max V(n) - \min U(n) \quad ,$$

$$\Delta_{U,U}(n) = \max U(n) - \min U(n) \quad ,$$

$$\Delta_{V,V}(n) = \max V(n) - \min V(n) \quad ,$$

and note the trivial identity

$$(2.8) \qquad \Delta_{U,V}(n) + \Delta_{V,U}(n) = \Delta_{U,U}(n) + \Delta_{V,V}(n) \quad .$$

We next introduce the vectors

$$\hat{U}(n) = U(n) - (\max U(0))1_r \quad ,$$

$$\hat{V}(n) = V(n) - (\min V(0))1_s \quad ,$$

where 1_r and 1_s are vectors with all components 1, and of dimensions r and s rexpectively. We note also that

$$(2.9) \qquad \max \hat{U}(0) = \min \hat{V}(0) = 0 \quad ,$$

and

$$(2.10) \qquad \Delta_{U,V}(n) - \Delta_{\hat{U},\hat{V}}(n) = \Delta_{U,V}(0) \quad .$$

<u>Lemma 2.2</u>. If $\max U(0) = \min V(0) = 0$, then $\Delta_{V,U}(n) \geq 0$.

Proof: Letting $y_k(n) = \sigma_k(n)/n$, $x_k(n) = \tau_k(n)/n$, we have from (2.5), (2.6),

$$\frac{\min U(n)}{n} \leq \frac{\max U(0)}{n} + \min \left(\sum_{k=1}^{s} y_k(n) C_k\right) \leq v \quad ,$$

and

$$\frac{\max V(n)}{n} \geq \frac{\min V(0)}{n} + \max \left(\sum_{k=1}^{r} x_k(n) R_k\right) \geq v \quad ,$$

and the lemma follows.

From (2.8) we see that under the assumption of the above lemma we have

$$\Delta_{U,V}(n) \le \Delta_{U,U}(n) + \Delta_{V,V}(n) \quad , \tag{2.11}$$

whereas from Lemma 2.1 we have (in all cases)

$$\Delta_{U,V}(n) \ge -\Delta_{V,U}(0) \quad . \tag{2.12}$$

Also, under the assumption of Lemma 2.2 we see that for

$a = \max\limits_{i,j} |a_{ij}|$,

$$\Delta_{U,V}(n) \le 2an \quad . \tag{2.13}$$

In carrying out the details of our estimate of the rate of convergence a simple numerical inequality will be needed, which we now provide as

Lemma 2.3. If $\overset{r+s}{m+n} \ge 4$ and $n \ge 2^{(r+s-1)(r+s-2)}$, then

$$\left[2^{r+s-3} - 2^{\frac{(r+s-3)^2}{r+s-2}}\right] n^{\frac{r+s-3}{r+s-2}} \ge 1 \quad .$$

Proof:

$$1 > \frac{1}{\sqrt{2}} + \frac{1}{16} \ge \frac{1}{2^{\frac{r+s-3}{r+s-2}}} + \frac{1}{2^{(r+s)(r+s-3)}}$$

so that

$$1 < \left(1 - 2^{-\frac{(r+s-3)}{r+s-2}}\right) 2^{(r+s)(r+s-3)}$$

$$= \left(2^{(r+s-3)} - 2^{\frac{(r+s-3)^2}{r+s-2}}\right) 2^{(r+s-1)(r+s-3)}$$

$$\leq \left(2^{(r+s-3)} - 2^{\frac{(r+s-3)^2}{r+s-2}}\right) n^{\frac{r+s-3}{r+s-2}} .$$

3. <u>Estimation of the rate of convergence</u>. The estimation of the rate of convergence focuses on ascertaining how fast $\Delta_{U,V}(n)/n$ goes to 0 as n goes to infinity. Since from (2.12) we have

$$\Delta_{U,V}(n)/n \geq -\Delta_{V,U}(0)/n \quad , \tag{3.1}$$

we need only focus on obtaining an upper estimate for this ratio. For this we have

<u>Theorem 3.1</u>. If $\max U(0) = \min V(0) = 0$ we have

$$\Delta_{U,V}(n)/n \leq \frac{a\ 2^{r+s}}{n^{\frac{1}{r+s-2}}} \quad , \tag{3.2}$$

and in general

$$\Delta_{U,V}(n)/n \leq \frac{a\ 2^{r+s}}{n^{\frac{1}{r+s-2}}} + \Delta_{U,V}(0)/n \quad , \tag{3.3}$$

where $a = \max_{i,j} |a_{ij}|$.

Proof: From (2.9) and (2.10) we note that (3.2) implies (3.3) so that we need only derive (3.2). This will be achieved by induction on r+s.

For r+s = 2, the matrix $A = (\alpha)$ is 1×1, so that if $U(0) = V(0) = 0$, $U(n)/n = V(n)/n = \alpha$. Thus in this case

$\triangle_{U,V}(n)/n = 0$, and (3.2) certainly true. We now assume that if $\max U(0) = \min V(0) = 0$, (3.2) holds for all matrices for which $r+s < k$, $k \geq 3$; and consider the case $r+s = k$.

Next we shall prove that given integers n and T, $0 < T < n$, then <u>either</u>

$$\text{(i)} \quad \triangle_{U,V}(n) \leq 4aT$$

<u>or</u> Of

$$\text{(ii)} \quad \triangle_{U,V}(n) - \triangle_{U,V}(n-T) < a2^{r+s-1}\, T^{1-\frac{1}{r+s-3}}.$$

For suppose (i) were false; so that $\triangle_{U,V}(n) > 4aT$. Then, from (2.11) we obtain that either $\triangle_{U,U}(n) > 2aT$ or $\triangle_{V,V}(n) > 2aT$. We will proceed on the assumption that $\triangle_{U,U}(n) > 2aT$, (an entirely analogous argument covers the other alternative), so that

$$\max U(n) - \min U(n) > 2aT \quad . \tag{3.4}$$

Suppose that max U(n) occurs in the i-th component and min U(n) occurs in the j-th component. Then since a component cannot change by more than <u>a</u> in one step of the process, (3.4) implies

$$U_i(n-w) - U_j(n-w) > 0 \quad , \tag{3.5}$$

for all integers w, $0 \leq w \leq T$. Thus we see that the j-th component never occurs as a maximum in the T steps preceding the n-th step. Let $A^{(j)}$ be the matrix obtained from A by deleting the j-th row, and let $U^{(j)}(m)$ be the vector obtained from U(m) by deleting the j-th component. If we then apply the iteration scheme to $A^{(j)}$, taking as initial vectors $U^{(j)}(n-T)$ and $V(n-T)$,

we clearly have that the first T+1 vectors of the respective $U^{(j)}$ and V series are

$$U^{(j)}(n-w) \quad , \qquad w = T, T-1, \ldots, 0 \quad ;$$

$$V(n-w) \quad , \qquad w = T, T-1, \ldots, 0 \quad .$$

Setting

$$(3.6) \qquad \begin{cases} \hat{U}^{(j)}(n-w) = U^{(j)}(n-w) - [\max U^{(j)}(n-T)]1_{r-1} \quad , \\ \hat{V}(n-w) = V(n-w) - [\min V(n-T)]1_s \quad , \end{cases}$$

we may apply the inductive hypothesis to obtain

$$(3.7) \qquad \max \hat{U}^{(j)}(n) - \min \hat{V}(n) \leq a\, 2^{r+s-1}\, T^{1 - \frac{1}{r+s-3}} \quad .$$

From (3.6) and the manner in which j was chosen, it follows that the left side of (3.7) equals $\triangle_{U,V}(n) - \triangle_{U,V}(n-T)$ so that we obtain (ii).

If $n \leq 2^{(r+s-1)(r+s-2)}$, utilizing (2.13) we have

$$\triangle_{U,V}(n) \leq 2an \leq 2an^{\frac{1}{r+s-2}}\, n^{1 - \frac{1}{r+s-2}}$$

$$\leq 2a\, 2^{r+s-1}\, n^{1 - \frac{1}{r+s-2}} = a2^{r+s}\, n^{1 - \frac{1}{r+s-2}} \quad .$$

Thus in this case $\triangle_{U,V}(n)/n \leq \dfrac{a2^{r+s}}{n^{1/r+s-2}}$, and the induction is completed. We therefore need only consider the case wherein $n > 2^{(r+s-1)(r+s-2)}$.

For a fixed positive integer T, we call n an integer of the first kind if (i) is satisfied, otherwise we call n an integer

of the <u>second kind</u>. In view of (2.13), if $n \leq T$, then n is of the first kind.

Let $q = [n/T]$. Then among the integers $n, n-T, n-2T, \ldots, n-qT$, there are integers of the first kind (since $n-qT$ is of the first kind). Let $n-\tau T$ be the largest integer of this set which is of the first kind. Then

$$\Delta_{U,V}(n) = \sum_{i=1}^{\tau} \left\{\Delta_{U,V}(n-(i-1)T) - \Delta_{U,V}(n-iT)\right\} + \Delta_{U,V}(n-\tau T)$$

$$\leq (a\ 2^{r+s-1}\ T^{1-\frac{1}{r+s-3}})\tau + 4aT$$

$$\leq na\ 2^{r+s-1}\ T^{-\frac{1}{r+s-3}} + 4aT \quad , \qquad (\text{since } \tau \leq n/T) \quad .$$

Thus for all $n \geq T > 0$ we have

$$\Delta_{U,V}(n)/n \leq a(2^{r+s-1}\ T^{-\frac{1}{r+s-3}} + 4T/n) \quad . \tag{3.8}$$

Consider first the case wherein $r+s = 3$. The result has been established for $n \leq 2^{(r+s-1)(r+s-2)} = 4$. For $n > 4$, and $T = 2$, (3.8) becomes

$$\Delta_{U,V}(n)/n \leq 8a/n \quad ,$$

which is precisely (3.2) in this case.

We next consider $r+s \geq 4$. From Lemma 2.3 we obtain the existence of an integer $T = T(n)$ such that

$$2^{\frac{(r+s-3)^2}{r+s-2}}\ n^{\frac{r+s-3}{r+s-2}} \leq T(n) \leq 2^{r+s-3}\ n^{\frac{r+s-3}{r+s-2}} \quad .$$

Recalling that we are carrying the assumption $n > 2^{(r+s-1)(r+s-2)}$, it is easily verified that $T(n)$ satisfies

$$\text{(a)} \quad T(n) \leq n \quad ,$$

$$\text{(b)} \quad 2^{r+s-1}\, T(n)^{-\frac{1}{r+s-3}} \leq 4T(n)/n \quad ,$$

and

$$\text{(c)} \quad 8T(n)/n \leq 2^{r+s}\, n^{-\frac{1}{r+s-2}} \quad .$$

Choosing $T = T(n)$ in (3.8), by means of (a), (b), and (c) above, (3.2) follows, and the induction is completed.

In conclusion we note that Theorem 3.1 together with Lemma 2.1 implies that

$$\left| v - \frac{\max U(n)}{n} \right| < \frac{c}{n^{\frac{1}{r+s-2}}} \; ; \qquad \left| v - \frac{\min V(n)}{n} \right| < \frac{c}{n^{\frac{1}{r+s-2}}} \quad .$$

4. <u>The case of symmetric games</u>. In the case of symmetric games, for which the game matrix is skew symmetric the above scheme is considerably simplified by the fact that we may take $U(0) = -V(0)$ and need only write down one of the two sequences $U(n)$, $V(n)$. Thus for example, we need only consider the recurrence

$$V(n+1) = V(n) + R_{i(n)} \quad ,$$

where

$$V_{i(n)}(n) = \min V(n) \quad .$$

The result of the previous section then asserts that $\min V(n)/n$ tends to 0 like $c/n^{1/2m-2}$, for an m by m skew-symmetric matrix.

However, the method of proof given above provides a more rapid rate of convergence in this case — namely like $c/n^{1/m-1}$. This stems from the fact that in the argument which led to the alternative (ii) in Section 3, when we delete a row (or column) we may (because of the symmetry) delete the same column (or row) also.

Whether or not the estimates of rate of convergence obtained above are best possible for this iteration scheme is not known. In all probability they are not. In fact, there is some evidence that the convergence is faster if some of the rows or columns are innessential. This suggests the possibility that the above method could be elaborated upon so as to provide an estimate of the form (3.3) in which r and s would be the number of rows and columns respectively of the essential part of the game matrix.

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